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Constrained derivatives and equilibrium conditions in generalized geometric programming

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SUMMARY

This paper demonstrates that the "equilibrium conditions" of generalized geometric programming can be interpreted as constrained derivatives of a transform of the dual program to the generalized geometric programming primal. Thus, iterative procedures employing these conditions amount to direct solution of the necessary conditions for a local minimum of the transformed dual expressed in constrained derivative form under the constraint qualification of nonsingularity and nondegeneracy.

1. Introduction

In an earlier paper, Passy and Wilde [1] show that the generalized geometric programming primal problem:

Minimize $g_0(x)$

Subject to $\sigma_m[g_m(x)]^{\sigma_m} \leqslant 1\,, \quad m=1, \ldots, M$

$x > 0$

where $$g_m(x) = \sum_{t=1}^{T_m} \sigma_{mt} C_{mt} \prod_{n=1}^{\xi} x_n^{a_{mtn}}$$

and the σ_{mt}, σ_m have assigned values, either $+1$ or -1,

can under suitable conditions be solved by simultaneous solution of a set of coupled linear and nonlinear equations. In particular, if a strictly positive solution ω_{mt}, $m=1, \ldots, M$; $t=1, \ldots, T_m$ to the following $N+1$ linear equations

$$\sum_{t=1}^{T_0} \sigma_{0t}\omega_{0t} = \sigma_0\,,$$

$$\sum_{m=1}^{M} \sum_{t=1}^{T_m} \sigma_{mt}\omega_{mt} a_{mtn} = 0\,, \quad n=1, \ldots, N\,,$$

and the following $T-(N+1)$ nonlinear "equilibrium" conditions

$$\prod_{m=0}^{M} \prod_{t=1}^{T_m} \left[\frac{\omega_{mt}}{\omega_{m0}}\right]^{\sigma_{mt}\omega_{mt}} = \prod_{m=0}^{M} \prod_{t=1}^{T_m} [C_{mt}]^{\sigma_{mt}V_{mtj}}\,, \quad j=1, \ldots, T-(N+1)\,, \tag{1}$$

where the V_{mtj} are the $T-N-1$ homogeneous solutions of

$$\sum_{t=1}^{T_0} \sigma_{0t} V_{0tj} = 0\,, \tag{2}$$

$$\sum_{m=0}^{M} \sum_{t=1}^{T_m} \sigma_{mt} a_{mtn} V_{mtj} = 0\,, \quad j=1, 2, \ldots, T-N-1\,, \tag{3}$$

can be found, then a primal solution vector $\tilde{x}$ may be recovered directly. This is accomplished by solving the following equations: (which are linear in the logarithms of the x_n)

$$\sum_{n=1}^{N} a_{0tn} \ln x_n = \ln \left[\frac{\omega_{0t} d(\tilde{\omega})}{C_{0t}} \right], \quad t = 1, \ldots, T_0 , \tag{4}$$

$$\sum_{n=1}^{N} a_{mtn} \ln x_n = \ln \left[\frac{\omega_{mt}}{\omega_{m0} C_{mt}} \right], \quad t = 1, \ldots, T_m , \; m = 1, \ldots, M , \tag{5}$$

where

$$d(\tilde{\omega}) \equiv \sigma_0 \left[\prod_{m=0}^{M} \prod_{t=1}^{T_m} \left(\frac{C_{mt} \omega_{m0}}{\omega_{mt}} \right)^{\sigma_{mt} \omega_{mt}} \right]^{\sigma_0} ,$$

and

$$\omega_{m0} \equiv \sigma_m \sum_{t=1}^{T_m} \sigma_{mt} \omega_{mt} , \quad m = 1, \ldots, M . \tag{6}$$

This approach has been exploited in [1] and [4] to solve the primal problem using suitable numerical equation solving routines. The point of this note is to demonstrate that the above $T-N-1$ equilibrium conditions can in fact be interpreted as the constrained derivatives [5] of a transformed dual problem:

Minimize $f(\tilde{\omega}) = -\sigma_0 \ln \sigma_0 d(\tilde{\omega})$,

Subject to

$$\sum_{t=1}^{T_m} \sigma_{mt} \omega_{mt} - \sigma_m \omega_{m0} = 0 , \quad m = 0, 1, \ldots, M , \tag{7}$$

$$\sum_{m=0}^{M} \sum_{t=1}^{T_m} \sigma_{mt} a_{mtn} \omega_{mt} = 0 , \quad n = 1, \ldots, N , \tag{8}$$

$$\omega_{mt} \geqslant 0 , \quad t = 1, \ldots, T_m , \; m = 0, 1, \ldots, M ,$$

with $\omega_{00} \equiv 1$,

under the constraint qualification of non-degeneracy and nonsingularity [5]. As a consequence of this identification, it is apparent that the solution procedures of [1], [2], and [4] are all equivalent. However, because a constrained derivative construction can be employed under less restrictive constraint qualifications [3], it appears that the constrained derivative formulation [2] is to be preferred.

2. Analysis

Consider a nonlinear programming problem (NLP)

Minimize $f(x)$,

Subject to $g_k(x) = 0 , \quad k = 1, \ldots, K ,$

$x \geqslant 0 , \quad x \in R^N .$

A point $\tilde{x}^0$ satisfying the above constraints is said to be *nonsingular* if the gradients of the constraints at $\tilde{x}^0$ are linearly independent. The point is further said to be *nondegenerate* if the Jacobian of these constraint gradients can be partitioned into K linearly independent columns corresponding to strictly positive components of $\tilde{x}^0$.

Let J be the submatrix consisting of the K columns corresponding to the components of $\tilde{x}^0$. Let C be a matrix containing the remaining columns and let (s, d) be the corresponding partition of the components of the vector $\tilde{x}^0$. Following [5], necessary conditions for a nondegenerate, non-singular point $\tilde{x}^0$ to be a local minimum for NLP are that

(i) $\dfrac{\delta f}{\delta d} \equiv \dfrac{\partial f}{\partial d} - \dfrac{\partial f}{\partial s} J^{-1} C \geqslant 0$

and

(ii) $\dfrac{\delta f}{\delta d_k} d_k = 0 , \quad k = 1, \ldots, N-K .$

Note that if all $d_k > 0$, then, from complementary slackness condition (ii), it follows that $\delta f/\delta d \equiv 0$.

The above result can be applied immediately to the transformed dual problem, provided the required partial derivatives are evaluated.

To this end observe that

$$\sigma_0 \ln \sigma_0 d(\tilde{\omega}) = \sum_{m=0}^{M} \sum_{t=1}^{T_m} \sigma_{mt}\omega_{mt} \ln \left[\frac{C_{mt}\omega_{m0}}{\omega_{mt}}\right].$$

Consequently, for any ω_{mt}

$$\frac{\partial f(\tilde{\omega})}{\partial \omega_{mt}} = -\sigma_{mt} \ln \left[\frac{C_{mt}\omega_{m0}}{\omega_{mt}}\right] + \sigma_{mt}\omega_{mt} \left[\frac{\omega_{mt}}{C_{mt}\omega_{m0}}\right] \left\{\frac{\sigma_m \sigma_{mt} C_{mt}}{\omega_{mt}} - \frac{C_{mt}\omega_{m0}}{\omega_{mt}^2}\right\}$$
$$+ \sum_{\substack{j=1 \\ j\neq t}}^{T_m} \sigma_{mj}\omega_{mj} \left[\frac{\omega_{mj}}{C_{mj}\omega_{m0}}\right] \left[\frac{C_{mj}\sigma_m \sigma_{mt}}{\omega_{mj}}\right].$$

However, because of the definition of ω_{m0} in equations [2], all terms except the first cancel and thus

$$\frac{\partial f(\tilde{\omega})}{\partial \omega_{mt}} = -\sigma_{mt} \ln \left(\frac{C_{mt}\omega_{m0}}{\omega_{mt}}\right) = \sigma_{mt} \ln \left(\frac{\omega_{mt}}{C_{mt}\omega_{m0}}\right). \tag{9}$$

This result will now be exploited in the following manner.

Let $\tilde{\omega}$ be a vector with ordered components $(\omega_{01}, \omega_{02}, \ldots, \omega_{0T_0}, \omega_{11}, \omega_{12}, \ldots, \omega_{1T_1}, \omega_{21}, \ldots, \omega_{M,T_M})$. Let the matrix H consist of the coefficients of the equality constraints (7) with $m=0$, and (8). If H has maximal row-rank and if all $\omega_{mt} > 0$, then any partition (J, C) of H such that J is square and nonsingular is acceptable for use in the calculation of the constrained derivatives. Without loss of generality assume that the columns of J correspond to the first $N+1$ components of the vector $\tilde{\omega}$. Further, let I_σ be the diagonal matrix with diagonal elements σ_{mt} and let $g(\tilde{\omega})$ be the row vector consisting of elements $\ln[\omega_{mt}/(C_{mt}\omega_{m0})]$, all ordered as the components of $\tilde{\omega}$ are ordered. Let I be an identity matrix of dimension $T-N-1$.

The constrained derivative then becomes

$$\frac{\delta f}{\delta d} = \frac{\partial f}{\partial d} - \frac{\partial f}{\partial s} J^{-1} C = \left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial d}\right) \begin{pmatrix} -J^{-1}C \\ I \end{pmatrix} = g I_\sigma \begin{pmatrix} -J^{-1}C \\ I \end{pmatrix}.$$

Assuming that all ω_{mt} and all ω_{m0} are positive, then the necessary conditions simply require that $\partial f/\partial d = 0$ or,

$$\frac{\delta f}{\delta d} = g I_\sigma \begin{pmatrix} -J^{-1}C \\ I \end{pmatrix} = 0. \tag{10}$$

The claim is that these conditions are equivalent to equation (1). To see that this is true, observe that the columns of $\begin{pmatrix} -J^{-1}C \\ I \end{pmatrix}$ are linearly independent (because of the presence of the identity matrix) and orthogonal to (J, C) since

$$(J, C) \begin{pmatrix} -J^{-1}C \\ I \end{pmatrix} = -C + C = 0.$$

But the $N+1$ rows of (J, C) are just the coefficients of equations (2) and (3). Hence, the $T-N-1$ linearly independent columns of $\begin{pmatrix} -J^{-1}C \\ I \end{pmatrix}$ span the null space of (J, C) and thus are equivalent to the $T-N-1$ homogeneous solutions V_{mtj} of equations (2) and (3). This construction provides a convenient method of calculating the homogeneous solutions. The equivalence between (1) and (10) follows upon taking the natural logarithm of equation (10):

$$\sum_{m=0}^{M} \sum_{t=1}^{T_m} \sigma_{mt} V_{mtj} \ln \left(\frac{\omega_{mt}}{C_{mt}\omega_{m0}}\right) = 0, \qquad j = 1, \ldots, T-N-1.$$

Finally, note that if the ω_{mt} corresponding to the columns of J are positive, but some of the remaining ω_{mt} are zero, then the constrained derivative (9) may be positive for those components d_r which are zero. Hence equation (10) must be supplemented with the complementary slackness condition $d_r \delta f/\delta d_r = 0$.

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